

A parametrization of the abstract Ramsey theorem

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Abstract

We give a parametrization with perfect subsets of 2^∞ of the abstract Ramsey theorem (see [13]). Our main tool is an extension of the parametrized version of the combinatorial forcing developed in [11] and [13], used in [8] to obtain a parametrization of the abstract Ellentuck theorem. As one of the consequences, we obtain a parametrized version of the Hales-Jewett theorem. Finally, we conclude that the family of perfectly \mathcal{S} -Ramsey subsets of $2^\infty \times \mathcal{R}$ is closed under the Souslin operation.

Key words and phrases: Ramsey theorem, Ramsey space, parametrization.

1 Introduction

In [13], S. Todorcevic presents an abstract characterization of those topological spaces in which an analog of Ellentuck's theorem [4] can be proven. These are called *topological Ramsey spaces* and the main result about them is referred to in [13] as *abstract Ellentuck theorem*. In [8], a parametrization with perfect subsets of 2^∞ of the abstract Ellentuck theorem is given, obtaining in this way new proofs of parametrized versions of the Galvin-Prikry theorem [6] (see [9]) and of Ellentuck's theorem (see [12]), as well as a parametrized version of Milliken's theorem [10]. The methods used in [8] are inspired by those used

in [5] to obtain a parametrization of the semiselective version of Ellentuck's theorem.

Nevertheless, topological Ramsey spaces are a particular kind of a more general type of spaces (introduced in [13]), in which the Ramsey property can be characterized in terms of the abstract Baire property. These are called *Ramsey spaces*. One of such spaces, known as the *Hales-Jewett space*, is described below (for a more complete description of this – non topological – Ramsey space, see [13]). S. Todorcevic has given a characterization of Ramsey spaces which is summed up in a result known as the *abstract Ramsey theorem*. It turns out that the abstract Ellentuck theorem is a consequence of the abstract Ramsey theorem (see [13]). Definitions of all these concepts will be given below.

In this work we adapt in a natural way the methods used in [8] in order to obtain a parametrized version of the abstract Ramsey theorem. In this way, we not only generalize the results obtained in [8] but we also obtain, in corollary 1 below, a parametrization of the infinite dimensional version of the Hales-Jewett theorem [7] (see [13]), which is the analog to Ellentuck's theorem corresponding to the Hales-Jewett space.

In the next section we summarize the definitions and main results related to Ramsey spaces given in [13]. In section 3 we introduce the (parametrized) combinatorial forcing adapted to the context of Ramsey spaces and present our main result (theorem 5 below). Finally, we conclude that the generalization of the *perfectly Ramsey property* (see [2] and [12]) to the context of Ramsey spaces is preserved by the Souslin operation (see corollary 4 below).

We'll use the following definitions and results concerning to perfect sets and trees (see [12]). For $x = (x_n)_n \in 2^\infty$, $x|_k = (x_0, x_1, \dots, x_{k-1})$. For $u \in 2^{infty}$, let $[u] = \{x \in 2^\infty : (\exists k)(u = x|_k)\}$ and denote the *length* of u by $|u|$. If $Q \subseteq 2^\infty$ is a perfect set, we denote T_Q its associated perfect tree. For $u, v = (v_0, \dots, v_{|v|-1}) \in 2^{<\infty}$, we write $u \sqsubseteq v$ to mean $(\exists k \leq |v|)(u = (v_0, v_1, \dots, v_{k-1}))$. Given $u \in 2^{<\infty}$, let $Q(u) = Q \cap [u(Q)]$, where $u(Q)$ is defined as follows: $\emptyset(Q) = \emptyset$. If $u(Q)$ is already defined, find $\sigma \in T_Q$ such that σ is the \sqsubseteq -extension of $u(Q)$ where the first ramification occurs. Then, set $(u \hat{\ } i)(Q) = \sigma \hat{\ } i$, $i = 1, 0$. Where " $\hat{\ }$ " is concatenation. Thus, for each n , $Q = \bigcup \{Q(u) : u \in 2^n\}$. For $n \in \mathbb{N}$ and perfect sets S, Q , we write $S \subseteq_n Q$ to mean $S(u) \subseteq Q(u)$ for every $u \in 2^n$. Thus " \subseteq_n " is a partial order and, if we have chosen $S_u \subseteq Q(u)$ for every $u \in 2^n$, then $S = \bigcup_u S_u$ is perfect, $S(u) = S_u$ and $S \subseteq_n Q$. The *property of fusion* of this order is: if $Q_{n+1} \subseteq_{n+1} Q_n$ for $n \in \mathbb{N}$, then $Q = \bigcap_n Q_n$ is perfect and $Q \subseteq_n Q_n$ for each n .

2 Abstract Ramsey theory

We introduce some definitions and results due to Todorcevic (see [13]). Our objects will be structures of the form $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)$ where \leq and \leq^0 are relations on $\mathcal{S} \times \mathcal{S}$ and $\mathcal{R} \times \mathcal{S}$ respectively; and r, s give finite approximations:

$$r: \mathcal{R} \times \omega \rightarrow \mathcal{AR} \quad s: \mathcal{S} \times \omega \rightarrow \mathcal{AS}$$

we denote $r_n(A) = r(A, n)$, $s_n(X) = s(X, n)$, for $A \in \mathcal{R}$, $X \in \mathcal{S}$, $n \in \mathbb{N}$. The following three axioms are assumed for every $(\mathcal{P}, p) \in \{(\mathcal{R}, r), (\mathcal{S}, s)\}$.

$$(A.1) \quad p_0(P) = p_0(Q), \text{ for all } P, Q \in \mathcal{P}.$$

$$(A.2) \quad P \neq Q \Rightarrow p_n(P) \neq p_n(Q) \text{ for some } n \in \mathbb{N}.$$

$$(A.3) \quad p_n(P) = p_m(Q) \Rightarrow n = m \text{ and } p_k(P) = p_k(Q) \text{ if } k < n.$$

In this way we can consider elements of \mathcal{R} and \mathcal{S} as infinite sequences $(r_n(A))_{n \in \mathbb{N}}$, $(s_n(X))_{n \in \mathbb{N}}$. Also, if $a \in \mathcal{AR}$ and $x \in \mathcal{AS}$ we can think of a and x as finite sequences $(r_k(A))_{k < n}$, $(s_k(X))_{k < m}$ respectively; with n, m the unique integers such that $r_n(A) = a$ and $s_m(X) = x$. Such n and m are called the *length* of a and the *length* of x , which we denote $|a|$ and $|x|$, respectively.

We say that $b \in \mathcal{AR}$ is an end-extension of $a \in \mathcal{AR}$ and write $a \sqsubseteq b$, if $(\exists B \in \mathcal{R} b = r_n(B)) \Rightarrow \exists m \leq |b| (a = r_m(B))$. In an analogous way we define the relation \sqsubseteq on \mathcal{AS} .

$$(A.4) \quad \textbf{Finitization:} \text{ There are relations } \leq_{fin} \text{ and } \leq_{fin}^0 \text{ on } \mathcal{AS} \times \mathcal{AS} \text{ and } \mathcal{AR} \times \mathcal{AS}, \text{ respectively, such that:}$$

- (1) $\{a: a \leq_{fin}^0 x\}$ and $\{y: y \leq_{fin} x\}$ are finite for all $x \in \mathcal{AS}$.
- (2) $X \leq Y$ iff $\forall n \exists m s_n(X) \leq_{fin} s_m(Y)$.
- (3) $A \leq^0 X$ iff $\forall n \exists m r_n(A) \leq_{fin}^0 s_m(X)$.
- (4) $\forall a \in \mathcal{AR} \forall x, y \in \mathcal{AS} [a \leq_{fin}^0 x \leq_{fin} y \Rightarrow (a \leq_{fin}^0 y)]$.
- (5) $\forall a, b \in \mathcal{AR} \forall x \in \mathcal{AS} [a \sqsubseteq b \text{ and } b \leq_{fin}^0 x \Rightarrow \exists y \sqsubseteq x (a \leq_{fin}^0 y)]$.

We deal with the *basic sets*

$$[a, Y] = \{A \in \mathcal{R}: A \leq^0 Y \text{ and } \exists n (r_n(A) = a)\}$$

$$[x, Y] = \{X \in \mathcal{S}: X \leq Y \text{ and } \exists n (s_n(X) = x)\}$$

for $a \in \mathcal{AR}$, $x \in \mathcal{AS}$ and $Y \in \mathcal{S}$. Notation:

$$[n, Y] = [s_n(Y), Y]$$

Also, we define the *depth* of $a \in \mathcal{AR}$ in $Y \in \mathcal{S}$ by

$$\text{depth}_Y(a) = \begin{cases} \min\{k: a \leq_{fin}^0 s_k(Y)\}, & \text{if } \exists k (a \leq_{fin}^0 s_k(Y)) \\ -1, & \text{otherwise} \end{cases}$$

The next result is immediate.

Lemma 1. *If $a \sqsubseteq b$ then $\text{depth}_Y(a) \leq \text{depth}_Y(b)$. ■*

Now we state the last two axioms:

(A.5) **Amalgamation:** $\forall a \in \mathcal{AR}, \forall Y \in \mathcal{S}$, if $\text{depth}_Y(a) = d$, then:

- (1) $d \geq 0 \Rightarrow \forall X \in [d, Y] ([a, X] \neq \emptyset)$.
- (2) Given $X \in \mathcal{S}$,

$$(X \leq Y \text{ and } [a, X] \neq \emptyset) \Rightarrow \exists Y' \in [d, Y] ([a, Y'] \subseteq [a, X])$$

(A.6) **Pigeon hole principle:** Suppose $a \in \mathcal{AR}$ has length l and $\mathcal{O} \subseteq \mathcal{AR}_{l+1} = r_{l+1}(\mathcal{R})$. Then for every $Y \in \mathcal{S}$ with $[a, Y] \neq \emptyset$, there exists $X \in [\text{depth}_Y(a), Y]$ such that $r_{l+1}([a, X]) \subseteq \mathcal{O}$ or $r_{l+1}([a, X]) \subseteq \mathcal{O}^c$.

Definition 1. We say that $\mathcal{X} \subseteq \mathcal{R}$ is \mathcal{S} -Ramsey if for every $[a, Y]$ there exists $X \in [\text{depth}_Y(a), Y]$ such that $[a, X] \subseteq \mathcal{X}$ or $[a, X] \subseteq \mathcal{X}^c$. If for every $[a, Y] \neq \emptyset$ there exists $X \in [\text{depth}_Y(a), Y]$ such that $[a, X] \subseteq \mathcal{X}^c$, we say that \mathcal{X} is \mathcal{S} -Ramsey null.

Definition 2. We say that $\mathcal{X} \subseteq \mathcal{R}$ is \mathcal{S} -Baire if for every $[a, Y] \neq \emptyset$ there exists a nonempty $[b, X] \subseteq [a, Y]$ such that $[b, X] \subseteq \mathcal{X}$ or $[b, X] \subseteq \mathcal{X}^c$. If for every $[a, Y] \neq \emptyset$ there exists a nonempty $[b, X] \subseteq [a, Y]$ such that $[b, X] \subseteq \mathcal{X}^c$, we say that \mathcal{X} is \mathcal{S} -meager.

It is clear that every \mathcal{S} -Ramsey set is \mathcal{S} -Baire and every \mathcal{S} -Ramsey null set is \mathcal{S} -meager.

Considering \mathcal{AS} with the discrete topology and $\mathcal{AS}^{\mathbb{N}}$ with the completely metrizable product topology; we say that \mathcal{S} is *closed* if it corresponds to a closed subset of $\mathcal{AS}^{\mathbb{N}}$ via the identification $X \rightarrow (s_n(X))_{n \in \mathbb{N}}$.

Definition 3. We say that $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)$ is a *Ramsey space* if every \mathcal{S} -Baire subset of \mathcal{R} is \mathcal{S} -Ramsey and every \mathcal{S} -meager subset of \mathcal{R} is \mathcal{S} -Ramsey null.

Theorem 1 (Abstract Ramsey theorem). *Suppose $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)$ satisfies (A.1) ... (A.6) and \mathcal{S} is closed. Then ■*

Example: The Hales-Jewett space

Fix a countable alphabet $L = \cup_{n \in \mathbb{N}} L_n$ with $L_n \subseteq L_{n+1}$ and L_n finite for all n ; fix $v \notin L$ a "variable" and denote W_L and W_{Lv} the semigroups of words over L and of variable words over L , respectively. Given $X = (x_n)_{n \in \mathbb{N}} \subseteq W_L \cup W_{Lv}$, we say that X is *rapidly increasing* if

$$|x_n| > \sum_{i=0}^{n-1} |x_i|$$

for all $n \in \mathbb{N}$. Put

$$W_L^{[\infty]} = \{X = (x_n)_{n \in \mathbb{N}} \subseteq W_L : X \text{ is rapidly increasing} \}$$

$$W_{Lv}^{[\infty]} = \{X = (x_n)_{n \in \mathbb{N}} \subseteq W_{Lv} : X \text{ is rapidly increasing} \}$$

By restricting to finite sequences with

$$r_n : W_L^{[\infty]} \rightarrow W_L^{[n]} \quad s_n : W_{Lv}^{[\infty]} \rightarrow W_{Lv}^{[n]}$$

we have rapidly increasing finite sequences of words or variable words. The *combinatorial subspaces* are defined for every $X \in W_{Lv}^{[\infty]}$ by

$$[X]_L = \{x_n[\lambda_0] \frown \cdots \frown x_{n_k}[\lambda_k] \in W_L : n_0 < \cdots < n_k, \lambda_i \in L_{n_i}\}$$

$$[X]_{Lv} = \{x_n[\lambda_0] \frown \cdots \frown x_{n_k}[\lambda_k] \in W_{Lv} : n_0 < \cdots < n_k, \lambda_i \in L_{n_i} \cup \{v\}\}$$

where " \frown " denotes concatenation of words and $x[\lambda]$ is the evaluation of the variable word x on the letter λ .

For $w \in [X]_L \cup [X]_{Lv}$ we call *support of w in X* the unique set $\text{supp}_X(w) = \{n_0 < n_1 < \cdots < n_k\}$ such that $w = x_{n_0}[\lambda_0] \frown \cdots \frown x_{n_k}[\lambda_k]$ as in the definition of the combinatorial subspaces $[X]_L$ and $[X]_{Lv}$. We say that $Y = (y_n)_{n \in \mathbb{N}} \in W_{Lv}^{[\infty]}$ is a *block subsequence* of $X = (x_n)_{n \in \mathbb{N}} \in W_{Lv}^{[\infty]}$ if $y_n \in [X]_{Lv} \forall n$ and

$$\max(\text{supp}_X(y_n)) < \min(\text{supp}_X(y_m))$$

whenever $n < m$, and write $Y \leq X$. We define the relation \leq^0 on $W_L^{[\infty]} \times W_{Lv}^{[\infty]}$ in the natural way. Then, if $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s) = (W_L^{[\infty]}, W_{Lv}^{[\infty]}, \leq, \leq^0, r, s)$ is as before, where r, s are the restrictions

$$r_n(X) = (x_0, x_1, \dots, x_{n-1}) \quad s_n(Y) = (y_0, y_1, \dots, y_{n-1})$$

we have (A.1) ... (A.6), particularly, (A.6) is the well known result:

Theorem 2. *For every finite coloring of $W_L \cup W_{Lv}$ and every $Y \in W_{Lv}^{[\infty]}$ there exists $X \leq Y$ in $W_{Lv}^{[\infty]}$ such that $[X]_L$ and $[X]_{Lv}$ are monochromatic. ■*

And as a particular case of theorem 1, we have (see [7])

Theorem 3 (Hales–Jewett). *The field of $W_{Lv}^{[\infty]}$ -Ramsey subsets of $W_L^{[\infty]}$ is closed under the Souslin operation and it coincides with the field of $W_{Lv}^{[\infty]}$ -Baire subsets of $W_L^{[\infty]}$. Moreover, the ideals of $W_{Lv}^{[\infty]}$ -Ramsey null subsets of $W_L^{[\infty]}$ and $W_{Lv}^{[\infty]}$ -meager subsets of $W_L^{[\infty]}$ are σ -ideals and they also coincide. ■*

3 The parametrization

We will denote the family of perfect subsets of 2^∞ by \mathbb{P} and define

$$\mathcal{AR}[X] = \{b \in \mathcal{AR} : [b, X] \neq \emptyset\}$$

also we'll use this notation

$$M \in \mathbb{P} \restriction Q \Leftrightarrow (M \in \mathbb{P}) \wedge (M \subseteq Q)$$

From now on we assume that $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)$ is an Ramsey space; that is, (A.1) ... (A.6) hold and \mathcal{S} is closed. The following are the abstract versions of *perfectly-Ramsey* sets and the $\mathbb{P} \times \text{Exp}(\mathcal{R})$ -Baire property as defined in [8].

Definition 4. $\Lambda \subseteq 2^\infty \times \mathcal{R}$ is *perfectly \mathcal{S} -Ramsey* if for every $Q \in \mathbb{P}$ and $[a, Y] \neq \emptyset$, there exist $M \in \mathbb{P} \restriction Q$ and $X \in [\text{depth}_Y(a), Y]$ with $[a, X] \neq \emptyset$ such that $M \times [a, X] \subseteq \Lambda$ or $M \times [a, X] \subseteq \Lambda^c$. If for every $Q \in \mathbb{P}$ and $[a, Y] \neq \emptyset$, there exist $M \in \mathbb{P} \restriction Q$ and $X \in [\text{depth}_Y(a), Y]$ with $[a, X] \neq \emptyset$ such that $M \times [a, X] \subseteq \Lambda^c$; we say that Λ is *perfectly \mathcal{S} -Ramsey null*.

Definition 5. $\Lambda \subseteq 2^\infty \times \mathcal{R}$ is *perfectly \mathcal{S} -Baire* if for every $Q \in \mathbb{P}$ and $[a, Y] \neq \emptyset$, there exist $M \in \mathbb{P} \restriction Q$ and $[b, X] \subseteq [a, Y]$ such that $M \times [b, X] \subseteq \Lambda$ or $M \times [b, X] \subseteq \Lambda^c$. If for every $Q \in \mathbb{P}$ and $[a, Y] \neq \emptyset$, there exist $M \in \mathbb{P} \restriction Q$ and $[b, X] \subseteq [a, Y]$ such that $M \times [b, X] \subseteq \Lambda^c$; we say that Λ is *perfectly \mathcal{S} -meager*.

Now, the natural extension of combinatorial forcing will be given. From now on fix $\mathcal{F} \subseteq 2^{<\infty} \times \mathcal{AR}$ and $\Lambda \subseteq 2^\infty \times \mathcal{R}$.

Combinatorial forcing 1 Given $Q \in \mathbb{P}$, $Y \in \mathcal{S}$ and $(u, a) \in 2^{<\infty} \times \mathcal{AR}[Y]$; we say that (Q, Y) *accepts* (u, a) if for every $x \in Q(u)$ and for every $B \in [a, Y]$ there exist integers k and m such that $(x|_k, r_m(B)) \in \mathcal{F}$.

Combinatorial forcing 2 Given $Q \in \mathbb{P}$, $Y \in \mathcal{S}$ and $(u, a) \in 2^{<\infty} \times \mathcal{AR}[Y]$; we say that (Q, Y) *accepts* (u, a) if $Q(u) \times [a, Y] \subseteq \Lambda$.

For both combinatorial forcings we say that (Q, Y) *rejects* (u, a) if for every $M \in \mathbb{P} \restriction Q(u)$ and for every $X \leq Y$ compatible with a ; (M, X) does not accept (u, a) . Also, we say that (Q, Y) *decides* (u, a) if it accepts or rejects it.

The following lemmas hold for both combinatorial forcings.

- Lemma 2.**
- a) If (Q, Y) *accepts (rejects)* (u, a) then (M, X) *also accepts (rejects)* (u, a) for every $M \in \mathbb{P} \restriction Q(u)$ and for every $X \leq Y$ compatible with a .
 - b) If (Q, Y) *accepts (rejects)* (u, a) then (Q, X) *also accepts (rejects)* (u, a) for every $X \leq Y$ compatible with a .
 - c) For all (u, a) and (Q, Y) with $[a, Y] \neq \emptyset$, there exist $M \in \mathbb{P} \restriction Q$ and $X \leq Y$ compatible with a , such that (M, X) *decides* (u, a) .
 - d) If (Q, Y) *accepts* (u, a) then (Q, Y) *accepts* (u, b) for every $b \in r_{|a|+1}([a, Y])$.
 - e) If (Q, Y) *rejects* (u, a) then there exists $X \in [\text{depth}_Y(a), Y]$ such that (Q, Y) *does not accept* (u, b) for every $b \in r_{|a|+1}([a, X])$.
 - f) (Q, Y) *accepts (rejects)* (u, a) iff (Q, Y) *accepts (rejects)* (v, a) for every $v \in 2^{<\infty}$ such that $u \sqsubseteq v$.

Proof: (a) and (b) follow from the inclusion: $M(u) \times [a, X] \subseteq Q(u) \times [a, Y]$ if $X \leq Y$ and $M \subseteq Q(u)$.

(c) Suppose that we have (Q, Y) such that for every $M \in \mathbb{P} \upharpoonright Q$ and every $X \leq Y$ compatible with a , (M, X) does not decide (u, a) . Then (M, X) does not accept (u, a) if $M \in \mathbb{P} \upharpoonright Q(u)$; i.e. (Q, Y) rejects (u, a) .

(d) Follows from: $a \sqsubseteq b$ and $[a, Y] \subseteq [b, Y]$, if $b \in r_{|a|+1}([a, X])$.

(e) Suppose (Q, Y) rejects (u, a) and define $\phi: \mathcal{AR}_{|a|+1} \rightarrow 2$ by $\phi(b) = 1$ if (Q, Y) accepts (u, b) . By (A.6) there exist $X \in [\text{depth}_Y(a), Y]$ such that ϕ is constant in $r_{|a|+1}([a, X])$. If $\phi(r_{|a|+1}([a, X])) = 1$ then (Q, X) accepts (u, a) , which contradicts (Q, Y) rejects (u, a) (by part (b)). The result follows.

(f) (\Leftarrow) Obvious.

(\Rightarrow) Follows from the inclusion: $Q(v) \subseteq Q(u)$ if $u \sqsubseteq v$. ■

We say that a sequence $([n_k, Y_k])_{k \in \mathbb{N}}$ is a *fusion sequence* if:

1. $(n_k)_{k \in \mathbb{N}}$ is nondecreasing and converges to ∞ .
2. $X_{k+1} \in [n_k, X_k]$ for all k .

Note that since \mathcal{S} is closed, for every fusion sequence $([n_k, Y_k])_{k \in \mathbb{N}}$ there exist a unique $Y \in \mathcal{S}$ such that $s_{n_k}(Y) = s_{n_k}(X_k)$ and $Y \in [n_k, X_k]$ for all k . Y is called the *fusion* of the sequence and is denoted $\lim_k X_k$.

Lemma 3. *Given $P \in \mathbb{P}$, $Y \in \mathcal{S}$ and $N \geq 0$; there exist $Q \in \mathbb{P} \upharpoonright P$ and $X \leq Y$ such that (Q, X) decides every $(u, a) \in 2^{<\infty} \times \mathcal{AR}[X]$ with $N \leq \text{depth}_X(a) \leq |u|$.*

Proof: We build sequences $(Q_k)_k$ and $(Y_k)_k$ such that:

1. $Q_0 = P$, $Y_0 = Y$.
2. $n_k = N + k$.
3. (Q_{k+1}, Y_{k+1}) decides every $(u, b) \in 2^{n_k} \times \mathcal{AR}[Y_k]$ with $\text{depth}_{Y_k}(b) = n_k$.

Suppose we have defined (Q_k, Y_k) . List $\{b_0, \dots, b_r\} = \{b \in \mathcal{AR}[Y_k] : \text{depth}_{Y_k}(b) = n_k\}$ and $\{u_0, \dots, u_{2^{n_k}-1}\} = 2^{n_k}$. By lemma 1(c) there exist $Q_k^{0,0} \in \mathbb{P} \upharpoonright Q_k(u_0)$ and $Y_k^{0,0} \in [n_k, Y_k]$ compatible with b_0 such that $(Q_k^{0,0}, Y_k^{0,0})$ decides (u_0, b_0) . In this way we can obtain $(Q_k^{i,j}, Y_k^{i,j})$ for every $(i, j) \in \{0, \dots, 2^{n_k}-1\} \times \{0, \dots, r\}$, which decides (u_i, b_j) and such that $Q_k^{i,j+1} \in \mathbb{P} \upharpoonright Q_k^{i,j}(u_i)$, $Y_k^{i,j+1} \leq Y_k^{i,j}$ is compatible with b_{j+1} , $Q_k^{i+1,0} \in \mathbb{P} \upharpoonright Q_k(u_{i+1})$ and $Y_k^{i+1,0} \leq Y_k^{i,r}$.

Define

$$Q_{k+1} = \bigcup_{i=0}^{2^{n_k}-1} Q_k^{i,r}, \quad Y_{k+1} = Y_k^{2^{n_k}-1,r}$$

Then, given $(u, b) \in 2^{n_k} \times \mathcal{AR}[Y_{k+1}]$ with $\text{depth}_{Y_{k+1}}(b) = n_k = \text{depth}_{Y_k}(b)$, there exist $(i, j) \in \{0, \dots, 2^{n_k} - 1\} \times \{0, \dots, r\}$ such that $u = u_i$ and $b = b_j$. So $(Q_k^{i,j}, Y_k^{i,j})$ decides (u, b) and, since

$$Q_{k+1}(u_i) = Q_k^{i,r} \subseteq Q_k^{i,j}(u_i) \subseteq Q_k^{i,j} \text{ and } Y_{k+1} \leq Y_k$$

we have (Q_{k+1}, Y_{k+1}) decides (u, b) (by lemma 1(a)). We claim that $Q = \bigcap_k Q_k$ and $X = \lim_k Y_k$ are as required: given $(u, a) \in 2^{<\infty} \times \mathcal{AR}[X]$ with $N \leq \text{depth}_X(a) \leq |u|$, we have $\text{depth}_X(a) = n_k = \text{depth}_{Y_k}(a)$ for some k . Then, if $|u| = n_k$, (Q_{k+1}, Y_{k+1}) from the construction of X decides (u, a) and hence (Q, X) decides (u, a) . If $|u| > n_k$, (Q, X) decides (u, a) by lemma 1(f). \blacksquare

Lemma 4. *Given $P \in \mathbb{P}$, $Y \in \mathcal{S}$, $(u, a) \in 2^{<\infty} \times \mathcal{AR}[Y]$ with $\text{depth}_Y(a) \leq |u|$ and (Q, X) as in lemma 2 with $N = \text{depth}_Y(a)$; if (Q, X) rejects (u, a) then there exist $Z \leq X$ such that (Q, Z) rejects (v, b) if $u \sqsubseteq v$, $a \sqsubseteq b$ and $\text{depth}_Z(b) \leq |v|$.*

Proof: Let's build a fusion sequence $([n_k, Z_k])_k$, with $n_k = |u| + k$. Let $Z_0 = X$. Then (Q, Z_0) rejects (u, a) (and by lemma 1(f) it rejects (v, a) if $u \sqsubseteq v$). Suppose we have (Q, Z_k) which rejects every (v, b) with $v \in 2^{n_k}$ extending u , $a \sqsubseteq b$ and $\text{depth}_{Z_k}(b) \leq n_k$. List $\{b_0, \dots, b_r\} = \{b \in \mathcal{AR}[Z_k] : a \sqsubseteq b \text{ and } \text{depth}_{Z_k}(b) \leq n_k\}$ and $\{u_0, \dots, u_s\}$ the set of all $v \in 2^{n_k+1}$ extending u . By lemma 1(f) (Q, Z_k) rejects (u_i, b_j) , for every $(i, j) \in \{0, \dots, s\} \times \{0, \dots, r\}$. Use lemma 1(e) to find $Z_k^{0,0} \in [n_k, Z_k]$ such that $(Q, Z_k^{0,0})$ rejects (u_0, b) if $b \in r_{|b_0|+1}([b_0, Z_k^{0,0}])$. In this way, for every $(i, j) \in \{0, \dots, s\} \times \{0, \dots, r\}$, we can find $Z_k^{i,j} \in [n_k, Z_k]$ such that $Z_k^{i,j+1} \in [n_k, Z_k^{i,j}]$, $Z_k^{i+1,0} \in [n_k, Z_k^{i,r}]$ and $(Q, Z_k^{i,j})$ rejects (u_i, b) if $b \in r_{|b_j|+1}([b_j, Z_k^{i,j}])$. Define $Z_{k+1} = Z_k^{s,r}$. Note that if $(v, b) \in 2^{<\infty} \times \mathcal{AR}[Z_{k+1}]$, $a \sqsubseteq b$, $u \sqsubseteq v$ and $\text{depth}_{Z_{k+1}}(b) = n_k + 1$ then $v = u_i$ for some $i \in \{0, \dots, s\}$ and $b = r_{|b|}(A)$, $a = r_{|a|}(A)$ for some $A \leq^0 Z_{k+1}$; by (A.4)(5) there exist $m \leq n_k$ such that $b' = r_{|b|-1}(A) \leq_{fin}^0 s_m(Z_{k+1})$, so $\text{depth}_{Z_{k+1}}(b') \leq n_k$, i.e. $b' = b_j$ for some $j \in \{0, \dots, r\}$. Then $b \in r_{|b_j|+1}([b_j, Z_k^{i,j}])$. Hence, by lemma 1(f), (Q, Z_{k+1}) rejects (v, b) . Then $Z = \lim_k Z_k$ is as required: given (v, b) with $u \sqsubseteq v$, $a \sqsubseteq b$ and $\text{depth}_Z(b) \leq |v|$ then $\text{depth}_Z(b) = \text{depth}_Y(a) + k \leq n_k$ for some k and $b \in r_{|b_j|+1}([b_j, Z_k^{i,j}])$ for some $j \in \{0, \dots, r\}$ from the construction of Z (again, by (A.4)(5)). So

(Q, Z_k) (from the construction of Z) rejects (v, b) and, by lemma 1(a), (Q, Z) also does it. ■

The following theorem is an extension of theorem 3 [8] and its proof is analogous.

Theorem 4. *For every $\mathcal{F} \subseteq 2^{<\omega} \times \mathcal{AR}$, $P \in \mathbb{P}$, $Y \in \mathcal{S}$ and $(u, a) \in 2^{<\omega} \times \mathcal{AR}$ there exist $Q \in \mathbb{P} \restriction P$ and $X \leq Y$ such that one of the following holds:*

1. *For every $x \in Q$ and $A \in [a, X]$ there exist integers $k, m > 0$ such that $(x_{|k}, r_m(A)) \in \mathcal{F}$.*
2. *$(T_Q \times \mathcal{AR}[X]) \cap \mathcal{F} = \emptyset$.*

Proof: Whitout loss of generality, we can assume $(u, a) = (\langle \rangle, \emptyset)$. Consider combinatorial forcing 1. Let (Q, X) as in lemma 3 ($N = 0$). If (Q, X) accepts $(\langle \rangle, \emptyset)$, part (1) holds. If (Q, X) rejects $(\langle \rangle, \emptyset)$, use lemma 4 to obtain $Z \leq X$ such that (Q, X) rejects $(u, a) \in 2^{<\omega} \times \mathcal{AR}[Z]$ if $\text{depth}_Z(b) \leq |u|$. If $(t, b) \in (T_Q \times \mathcal{AR}[Z]) \cap \mathcal{F}$, find $u_t \in 2^{<\omega}$ such that $Q(u_t) \subseteq Q \cap [t]$. Thus, (Q, Z) accepts (u, b) . In fact: for $x \in Q(u_t)$ and $B \in [b, Z]$ we have $(x_{|k}, r_m(A)) = (t, b) \in \mathcal{F}$ if $k = |t|$ and $m = |b|$. By lemma 2(f), (Q, Z) accepts (v, b) if $u_t \sqsubseteq v$ and $\text{depth}_Z(b) \leq |v|$. But this is a contradiction with the choice of Z . Hence, $(T_Q \times \mathcal{AR}[X]) \cap \mathcal{F} = \emptyset$. ■

The next theorem is our main result and its proof is analogous to theorem 3 [8].

Theorem 5. *For $\Lambda \subseteq 2^\omega \times \mathcal{R}$ we have:*

1. *Λ is perfectly \mathcal{S} -Ramsey iff it is perfectly \mathcal{S} -Baire.*
2. *Λ is perfectly \mathcal{S} -Ramsey null iff it is perfectly \mathcal{S} -meager.*

Proof: (1) We only have to prove the implication from right to left. Suppose that $\Lambda \subseteq 2^\omega \times \mathcal{R}$ is perfectly \mathcal{S} -Baire. Again, whitout loss of generality, we can lead whith a given $Q \times [\emptyset, Y]$. Using combinatorial forcing and lemma 3, we have the following:

Claim 1. *Given $\hat{\Lambda} \subseteq 2^\omega \times \mathcal{R}$, $P \in \mathbb{P}$ and $Y \in \mathcal{S}$, there exists $Q \in \mathbb{P} \restriction P$ and $X \leq Y$ such that for each $(u, b) \in 2^{<\omega} \times \mathcal{AR}[X]$ with $\text{depth}_X(b) \leq |u|$ one of the following holds:*

- i) $Q(u) \times [b, X] \subseteq \hat{\Lambda}$
- ii) $R \times [b, Z] \not\subseteq \hat{\Lambda}$ for every $R \subseteq Q(u)$ and every $Z \leq X$ compatible with b .

By applying the claim to Λ , P and Y , we find $Q_1 \in \mathbb{P} \restriction P$ and $X_1 \leq Y$ such that for each $(u, b) \in 2^{<\infty} \times \mathcal{AR}[X_1]$ with $\text{depth}_{X_1}(b) \leq |u|$ one of the following holds:

- $Q_1(u) \times [b, X_1] \subseteq \Lambda$ or
- $R \times [b, Z] \not\subseteq \Lambda$ for every $R \subseteq Q_1(u)$ and every $Z \leq X_1$ compatible with b .

For each $t \in T_{Q_1}$, choose $u_1^t \in 2^{<\infty}$ with $u_1^t(Q_1) \sqsubseteq t$. If we define the family

$$\mathcal{F}_1 = \{(t, b) \in T_{Q_1} \times \mathcal{AR}[X_1] : Q_1(u_1^t) \times [b, X_1] \subseteq \Lambda\}$$

then we find $S_1 \subseteq Q_1$ and $Z_1 \leq X_1$ as in theorem 4. If (1) of theorem 4 holds, we are done. If part (2) holds, apply the claim to Λ^c , S_1 and Z_1 to find $Q_2 \in \mathbb{P} \restriction P$ and $X_2 \leq Y$ such that for each $(u, b) \in 2^{<\infty} \times \mathcal{AR}[X_2]$ with $\text{depth}_{X_2}(b) \leq |u|$ one of the following holds:

- $Q_2(u) \times [b, X_2] \subseteq \Lambda^c$ or
- $R \times [b, Z] \not\subseteq \Lambda^c$ for every $R \subseteq Q_2(u)$ and every $Z \leq X_2$ compatible with b .

Again, for each $t \in T_{Q_2}$, choose $u_2^t \in 2^{<\infty}$ with $u_2^t(Q_2) \sqsubseteq t$; define the family

$$\mathcal{F}_2 = \{(t, b) \in T_{Q_2} \times \mathcal{AR}[X_2] : Q_2(u_2^t) \times [b, X_2] \subseteq \Lambda\}$$

and find $S_2 \subseteq Q_2$ and $Z_2 \leq X_2$ as in theorem 4. If (1) holds, we are done and part (2) is not possible since Λ is perfectly \mathcal{S} -Baire (see [8]). This proves (1). To see part (2), notice that, as before, we only have to prove the implication from right to left, which follows from part (1) if Λ is perfectly \mathcal{S} -meager. ■

Corollary 1 (Parametrized infinite dimensional Hales-Jewett theorem). *For $\Lambda \subseteq 2^\infty \times W_L^{[\infty]}$ we have:*

1. Λ is perfectly Ramsey iff it has the $\mathbb{P} \times W_{Lv}^{[\infty]}$ -Baire property.
2. Λ is perfectly Ramsey null iff it is $\mathbb{P} \times W_{Lv}^{[\infty]}$ -meager. ■

Making $\mathcal{R} = \mathcal{S}$ in $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)$, we obtain the following:

Corollary 2 (Mijares). *If $(\mathcal{R}, \leq, (p_n)_{n \in \mathbb{N}})$ is a topological Ramsey space then:*

1. $\Lambda \subseteq \mathcal{R}$ is perfectly Ramsey iff has the $\mathbb{P} \times \text{Exp}(\mathcal{R})$ -Baire property.

2. $\Lambda \subseteq \mathcal{R}$ is perfectly Ramsey null iff is $\mathbb{P} \times \text{Exp}(\mathcal{R})$ -meager. ■

Corollary 3 (Pawlikowski). For $\Delta \subseteq 2^\infty \times \mathbb{N}^{[\infty]}$ we have:

1. Λ is perfectly Ramsey iff it has the $\mathbb{P} \times \text{Exp}(\mathbb{N}^{[\infty]})$ -Baire property.
2. Λ is perfectly Ramsey null iff it is $\mathbb{P} \times \text{Exp}(\mathbb{N}^{[\infty]})$ -meager. ■

Now we will proof that the family of perfectly \mathcal{S} -Ramsey and perfectly \mathcal{S} -Ramsey null subsets of $2^\infty \times \mathcal{R}$ are closed under the Souslin operation. Recall that the result of applying the Souslin operation to a given $(\Lambda_a)_{a \in \mathcal{AR}}$ is

$$\bigcup_{A \in \mathcal{R}} \bigcap_{n \in \mathbb{N}} \Lambda_{r_n(A)}$$

Proposition 1. The perfectly \mathcal{S} -Ramsey null subsets of $2^\infty \times \mathcal{R}$ form a σ -ideal.

Proof: This proof is also analogous to its corresponding version in [8] (lemma 4). So we just expose the mean ideas. Given an increasing sequence of perfectly \mathcal{S} -Ramsey null subsets of $2^\infty \times \mathcal{R}$ and $P \times [\emptyset, Y]$, we proceed as in lemma 3 to build fusion sequences $(Q_n)_n$ and $[n+1, X_n]$ such that

$$Q_n \times [b, X_n] \cap \Lambda_n = \emptyset$$

for every $n \in \mathbb{N}$ and $b \in \mathcal{AR}[X_n]$ with $\text{depth}_{X_n}(b) \leq n$. Thus, if $Q = \bigcap_n Q_n$ and $X = \lim_n X_n$, we have $Q \times [\emptyset, X] \cap \bigcup_n \Lambda_n = \emptyset$. ■

Recall that given a set X , two subsets A, B of X are "compatibles" with respect to a family \mathcal{F} of subsets X if there exists $C \in \mathcal{F}$ such that $C \subseteq A \cap B$. And \mathcal{F} is M -like if for $\mathcal{G} \subseteq \mathcal{F}$ with $|\mathcal{G}| < |\mathcal{F}|$, every member of \mathcal{F} which is not compatible with any member of \mathcal{G} is compatible with $X \setminus \bigcup \mathcal{G}$. A σ -algebra \mathcal{A} of subsets of X together with a σ -ideal $\mathcal{A}_0 \subseteq \mathcal{A}$ is a *Marczewski pair* if for every $A \subseteq X$ there exists $\Phi(A) \in \mathcal{A}$ such that $A \subseteq \Phi(A)$ and for every $B \subseteq \Phi(A) \setminus A$, $B \in \mathcal{A} \rightarrow B \in \mathcal{A}_0$. The following is a well known fact:

Theorem 6 (Marczewski). Every σ -algebra of sets which together with a σ -ideal is a Marczewski pair, is closed under the Souslin operation. ■

Let's denote $\mathcal{E}(\mathcal{S}) = \{[n, Y] : n \in \mathbb{N}, Y \in \mathcal{S}\}$.

Proposition 2. If $|\mathcal{S}| = 2^{\aleph_0}$, then the family $\mathcal{E}(\mathcal{S})$ is M -like.

Proof: Consider $\mathcal{B} \subseteq \mathcal{E}(\mathcal{S})$ with $|\mathcal{B}| < |\mathcal{E}(\mathcal{S})| = 2^{\aleph_0}$ and suppose that $[a, Y]$ is not compatible with any member of \mathcal{B} , i. e. for every $B \in \mathcal{B}$, $B \cap [a, Y]$ does not contain any member of $\mathcal{E}(\mathcal{S})$. We claim that (Q, Y) is compatible with $\mathcal{R} \setminus \bigcup \mathcal{B}$. In fact:
 Since $|\mathcal{B}| < 2^{\aleph_0}$, $\bigcup \mathcal{B}$ is \mathcal{S} -Baire (it is \mathcal{S} -Ramsey). So, there exist $[b, X] \subseteq [a, Y]$ such that:

1. $[b, X] \subseteq \bigcup \mathcal{B}$ or
2. $[b, X] \subseteq \mathcal{R} \setminus \bigcup \mathcal{B}$

(1) is not possible because $[a, Y]$ is not compatible with any member of \mathcal{B} . And (2) says that $[a, Y]$ is compatible with $\mathcal{R} \setminus \bigcup \mathcal{B}$ ■

As consequences of the previous proposition and theorem 6, the following facts hold.

Corollary 4. *If $|\mathcal{S}| = 2^{\aleph_0}$, then the family of perfectly \mathcal{S} -Ramsey subsets of $2^\infty \times \mathcal{R}$ is closed under the Souslin operation.* ■

Corollary 5. *The field of perfectly $W_{L_v}^{[\infty]}$ -Ramsey subsets of $2^\infty \times W_L^{[\infty]}$ is closed under the Souslin operation.* ■

Finally, making $\mathcal{R} = \mathcal{S}$ in $(\mathcal{R}, \mathcal{S}, \leq, \leq^0, r, s)$, we obtain the following:

Corollary 6 (Mijares). *If (\mathcal{R}, \leq, r) satisfies (A.1) ... (A.6), \mathcal{R} is closed, and $|\mathcal{R}| = 2^{\aleph_0}$ then the family of perfectly Ramsey subsets of $2^\infty \times \mathcal{R}$ is closed under the Souslin operation.* ■

Corollary 7 (Pawlikowski). *The field of perfectly Ramsey subsets of $2^\infty \times \mathbb{N}^{[\infty]}$ is closed under the Souslin operation.* ■

References

- [1] Carlson, T. J, Simpson, S. G. *Topological Ramsey theory*, in Nešetřil, J., Rödl, *Mathematics of Ramsey Theory*(Eds.), Springer, Berlin, 1990, pp. 172–183.
- [2] Di Prisco, C., *Partition properties and perfect sets*, Adv. in Math., **176**(2003), 145–173.
- [3] Di Prisco, C., Todorcevic, S., *Souslin partitions of products of finite sets*, Notas de Lógica Matemática Vol. 8, Universidad Nacional del Sur, Bahía Blanca, Argentina, 1993, pp. 119-127.

- [4] Elentuck, E. *A new proof that analitic sets are Ramsey*, J. Symbolic Logic, **39**(1974), 163–165.
- [5] Farah, I. *Semiselective coideals*, Mathematika., **45**(1998), 79–103.
- [6] Galvin, F., Prikry, K. *Borel sets and Ramsey's theorem*, J. Symbolic Logic, **38**(1973), 193–198.
- [7] Hales, A.W. and Jewett, R.I., *Regularity and Propositional Games*, Trans. Amer. Math. **106** (1963), 222–229.
- [8] Mijares, J. *Parametrizing the abstract Ellentuck theorem*, Discrete Math., **307**(2007), 216–225.
- [9] A. Miller, *Infinite combinatorics and definibility*, Ann. Pure Appl. Logic **41**(1989), 178–203.
- [10] Milliken, K., *Ramsey's theorem with sums or unions*, J. Comb. Theory, ser A **18**(1975), 276–290.
- [11] Nash-Williams, C. St. J. A., *On well-quasi-ordering transfinite sequences*, Proc. Cambridge Philo. Soc., **61**(1965), 33–39.
- [12] J. Pawlikowski, *Parametrized Elletuck theorem*, Topology and its applications **37**(1990), 65–73.
- [13] Todorcevic, S., *Introduction to Ramsey spaces*, Princeton University Press, Princeton, New Jersey, 2010.